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HIGH FREQUENCY OSCILLATIONS OF A THIN ELASTIC LAYER OF VARIABLE THICKNESS

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I. A. MOLOTKOV and D. K. OZEROV
(Leningrad)
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We investigate shear oscillations of a thin elastic layer $0 \leq z \leq h(x)$ of variable thickness, where $h(x)$ is a sufficiently smooth function. One boundary of this layer is free, while the other is in contact with a nonhomogeneous elastic medium, the contact defined by a boundary condition containing an impedance. Oscillations are of high-frequency

$$\Omega \equiv \frac{\omega h(x)}{b} \gg 1$$

Here ω is the frequency and b denotes the rate of propagation of shear waves. The displacement vector is parallel to the Y-axis.

Solution of the problem is constructed in the form of special asymptotic power series in $\Omega^{-1/3}$. Displacements of the layer are expressed in terms of $h(x)$ and of the properties of the elastic medium in contact with the layer. Expressions are found for the phase and group velocity within the layer. Final formulas are also obtained by another method based on the idea of constructive interference of the volume waves. Radial interpretation of the dependence of wave intensity on the variables x and z and ray tracing method, are used to obtain the decay of perturbations propagating along the layer.

1. Let us consider a homogeneous elastic layer

$$0 \leqslant z \leqslant h(x) \tag{1.1}$$

in the Cartesian coordinate system.

Equations of the theory of elasticity (under a wide class of boundary conditions) allow us to isolate, from the general problem on oscillations of a layer (1.1), a scalar problem for displacements u parallel to the y-axis

$$\mathbf{u} = e^{-i\omega t}\mathbf{j}u(x, z, k), \qquad k = \omega b^{-1} \tag{1.2}$$

Lamé equation yields the Helmholtz equation

$$u_{xx} + u_{zz} + k^2 u = 0 \tag{1.3}$$

We assume that the boundary $z = 0$ is stress-free

$$u_z \big|_{z=0} = 0 \tag{1.4}$$

while on $z = h$ we have the following boundary condition with an impedance

$$u_z - ikg(x)\, u \,|_{z=h(x)} = 0, \qquad g(x) > 0 \tag{1.5}$$

which describes the interaction between the layer and a nonhomogeneous elastic medium situated in the region $z > h(x)$ and characterized by the wave propagation velocity equal to $b_1(x, z) < b$. When $\omega \to \infty$, we have

$$g(x) = \frac{\rho_1(x, z)\, b_1(x, z) \sqrt{b^2 - b_1^2(x, z)}}{\rho b^2} \bigg|_{z=h(x)} \tag{1.6}$$

where ρ and $\rho_1(x, z)$ denote densities of the medium contained in the layer (1.1), and the region $z > h(x)$, respectively. Functions $h(x)$ and $g(x)$ shall be assumed to be sufficiently smooth (e.g. functions and their second derivatives are both assumed to be continuous).

Let us consider the eigenfunctions of the problem (1.3) to (1.5) possessing a character of the waves travelling along the x-axis (Love waves)

$$u(x, z, k) = Ue^{ikx} \tag{1.7}$$

We shall seek these eigenfunctions in a high-frequency approximation, i.e. when

$$kh(x) \gg 1 \tag{1.8}$$

(thickness of the layer is much greater here, than the wavelength). Moreover, we shall study such solutions of the problem (1.3) to (1.5), which correspond to the rays reflecting alternately from both boundaries of the layer. We find that such an alternating reflection certainly takes place when

$$kh(x)\, h'(x) \ll 1 \tag{1.9}$$

i.e. when the thickness of the layer is small and varies with x so smoothly, that this variation is small compared with $(kh)^{-1}$. Inequalities (1.8) and (1.9) limit the frequency of oscillations under consideration both, from above and from below.

2. The problem stated above differs from the usual problem of determining the high-frequency asymptotics; we cannot say that the large parameter kh exceeds all the other dimensionless parameters of the problem by a large amount.

To construct the solution, we shall have to assume that

$$h(x) = k^{-p} f(x) \qquad (1/2 < p < 1) \tag{2.1}$$

with $f(x)$ assuming finite values. Then, conditions (1.8) and (1.9) will hold, provided that k is sufficiently large.

We shall also introduce an independent variable $\zeta = k^p z$. Then, from (1.7) we shall obtain the following problem for the function $U(x, \zeta, k)$:

$$U_{xx} + k^{2p} U_{\zeta\zeta} + 2ikU_x = 0, \quad U_\zeta|_{\zeta=0} = 0, \quad U_\zeta + ik^{1-p} g(x)\, U|_{\zeta=f(x)} = 0 \qquad (2.2)$$

which does not contain a large parameter of the same order as k, and competing with it. Asymptotic expansion of its solution (with $k \to \infty$ and an arbitrary, though fixed $f(x)$), shall be sought in the form

$$U(x, \zeta, k) = \exp[k^q \varphi(x) + \Lambda(x, \zeta, k)] \cos L(x, \zeta, k)$$

$$\Lambda \sim \alpha(x, \zeta) + \frac{\beta(x, \zeta)}{k^q} + \frac{\gamma(x, \zeta)}{k^{2q}} + \cdots \qquad (2.3)$$

$$L \sim a(x, \zeta) + \frac{b(x, \zeta)}{k^q} + \frac{c(x, \zeta)}{k^{2q}} + \cdots \qquad (0 < q < p)$$

Here $\varphi(x)$, $\alpha(x, \zeta)$, $a(x, \zeta)$, $\beta(x, \zeta)$ and $b(x, \zeta)$, ... are functions to be determined, while q and p are unknown values. Inserting (2.3) into (2.2) and equating to zero the coefficients of each power of k, we obtain a recurring system of differential equations and boundary conditions for the unknown functions. To secure the existence of nontrivial solutions of this system, we find it necessary to put $p = {}^2/_3$ and $q = {}^1/_3$.

Omitting the actual procedure of obtaining it, we shall now state the final result. Problem (2.2) possesses an enumerable set of solutions given by the following asymptotic formulas:

$$U_m(x, \zeta, k) = \frac{C_m}{\sqrt{f(x)}} \exp\left[-\frac{i(m - {}^1/_2)^2 \pi^2 k^{1/3}}{2} \int_0^x \frac{dx}{f^2(x)} - \left(m - \frac{1}{2}\right)^2 \pi^2 \int_0^x \frac{dx}{f^3(x)\, g(x)} + \right.$$
$$\left. + \frac{1}{k^{1/3}} \left(\frac{i}{2} \frac{f'(x)}{f(x)} \zeta^2 + \eta(x)\right) + O\left(\frac{1}{k^{2/3}}\right)\right] \cos\left[\frac{(m - {}^1/_2)\pi\zeta}{f(x)} \left(1 - \frac{i}{k^{1/3} f(x)\, g(x)}\right) + O\left(\frac{1}{k^{2/3}}\right)\right]$$
$$(m = 1, 2, 3, \ldots) \qquad (2.4)$$

where C_m is an arbitrary function of k, and

$$f(x) = k^{2/3} h(x), \qquad \zeta = k^{2/3} z$$

Choice of the point $x = 0$ is arbitrary. The function $\eta(x)$ will not be quoted here in full since it is comparatively bulky, and we shall just mention that functions $f(x)$, $g(x)$, $f'(x)$, $g'(x)$ and $f''(x)$ all enter into it.

Inserting (2.4) into (1.7) and (1.2) we obtain formulas for displacements corresponding to separate Love waves. A multiplier

$$\exp\left[-\left(m - \frac{1}{2}\right)^2 \pi^2 \int_0^x \frac{dx}{f^3(x)\, g(x)}\right] \qquad (2.5)$$

describes, in an approximate manner, the decay of Love waves caused by their interaction with the medium situated at $z > h(x)$.

Phase $v_1^{(m)}$ and group $v_2^{(m)}$ velocities of these waves on the basis of (2.2) are

$$v_{1,2}^{(m)}(x) = b \pm \frac{(m - {}^1/_2)^2 \pi^2 b^{5/3}}{2\omega^{2/3} f^2(x)} + O(\omega^{-4/3}) \qquad (m = 1, 2, 3, \ldots) \qquad (2.6)$$

Region of applicability of the formulas obtained can be deduced from the form of asymptotics of (2.2). When the functions $h(x)$ and $g(x)$ are smooth, the condition that (1.8) holds together with the inequalities

$$kh(x)\, h'(x) \ll m, \qquad kh(x) \gg mg^{-1}(x), \qquad kh(x) \gg m \qquad (2.7)$$

is sufficient.

Then Formulas (1.2), (1.7), (2.4) and (2.6) describe the behavior of Love waves propagating in the layer (1.1) with alternate reflections from its boundaries.

Similar asymptotic formulas can also be obtained for a layer with variable velocity $b = b(x, z)$. Results of [1] show clearly that in this case new solutions appear in addition to those of the form (2.4). The cosine multiplier is replaced, in these solutions, by a multiplier containing an Airy integral.

3. All essential parts of the above results can be obtained using the radial approach, which allows a clear interpretation of the process of propagation of the interference waves.

Relation (2.6) for the phase velocity can be obtained by the method of constructive interference [2 and 3]. We assume that energy, in the considered medium, is basically propagated along the rays, alternately reflecting from the boundaries $z = 0$ and $z = h$, where the phases of reflection coefficients are 0 and π, respectively. Moreover, in order to obtain Formula (2.6) for $v_1^{(m)}$, we must assume that

$$\operatorname{tg}\theta \operatorname{tg}\psi \ll 1 \tag{3.1}$$

$${}^1/{}_2\pi - \theta \ll 1 \tag{3.2}$$

where θ is the angle of incidence of the ray on the boundary, while ψ is the angle between the tangent to the boundary $z = h(x)$ and the x-axis. Condition (3.1) is equivalent to the first inequality of (2.7). It expresses the requirement that the thickness of the layer varies, over the distance Δ between the two neighboring points of incidence of the ray, by an amount which is small compared with $h(x)$ i.e. $h'(x)\Delta \ll h(x)$. With this condition fulfilled, $h'(x)$ does not influence the principal terms of the expression for $v_1^{(m)}$ Inequality (3.2) infers that phase velocities of the waves under consideration, differ little from the velocity of shear waves in the layer, and is equivalent to the third inequality of (2.7).

The same method can be used to obtain the formula for group velocity (with the horizontal distance Δ divided by the time of passage along the ray).

Principal terms of (2.4) can be obtained by summation of waves of two types, of these moving towards the boundary $z = 0$ with those moving towards $z = h(x)$. If we neglect the absorption on the second boundary (this corresponds to $g(x) \to \infty$), then the summation will yield the factor $\cos[\pi(m - {}^1/{}_2)\zeta f^{-1}(x)]$ which is the fundamental ζ-dependent factor appearing in the right-hand side of (2.4).

Appearance of the factor $f^{-1/2}(x)$ in (2.4) is related to an obvious increase (decrease) in the energy density when the thickness of the layer decreases (increases).

Let us now consider the exponential decay of the Love waves along the layer. Boundary condition with an impedance (1.5) has a corresponding coefficient of reflection of a plane wave

$$\varkappa(x) = [\cos\theta - g(x)] / [\cos\theta + g(x)]$$

which becomes

$$\varkappa(x) \approx -1 + \frac{2\cos\theta}{g(x)} \tag{3.3}$$

when $\cos\theta / g(x) \ll 1$ the latter condition being equivalent to the second inequality of (2.7).

Such a reflection coefficient occurs e.g. when a layer moving with an increased velocity is in a rigid contact with a semi-space moving at a reduced velocity. But then Formula (1.6) is valid for $g(x)$.

Formula (3.3) implies that when a wave of unit amplitude is reflected from the boundary $z = h(x)$, then the amplitude diminishes by a small amount equal to $2\cos\theta g^{-1}(x)$. Obviously, the field of such a wave should vary smoothly along the layer. Then, the unit amplitude will be reduced, over the unit distance in the x-direction, by

$$\cos^2 \theta \,/\, h(x)\, g(x)$$

since $\Delta = 2h(x)\operatorname{tg}\theta$. Assuming that the amplitude decrement is proportional to its absolute magnitude and to the distance spanned along the x-axis, we obtain, by elementary integration, the factor (2.5) defining the decay.

Taking now into account the influence of variation in amplitude in the x-direction on the variation of amplitude in the z-direction, we obtain a more accurate expression involving the change in intensity with depth within the layer, and we shall find that this expression is identical with the complete cosine multiplier appearing in (2.4).

We see therefore, that the radial approach also yields the principal terms of (2.4) and (2.6) under the same assumptions concerning the properties of the medium and the frequency of oscillations.

Results obtained by us are applicable, primarily, in geophysics. For example, they provide a theoretical justification for use of a widely employed method of determination of variable thickness of the Earth's crust by utilizing localized values of phase velocities of the Love waves. They may also be of interest in mechanics of thin layers and plates, in connection with still increasing importance of the impulsive and high frequency effects.

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